DE RHAM THEOREM

The de Rham theorem brings together simplicial homology and de Rham-cohomology.

We have the $\mathbb{Z}$-module of singular k-simplices on M $C_k(M)$ and the vectorspace of k-forms on M $\Omega^k(M)$.
For a k-form $\omega \in \Omega^k(M)$ and a k-chain $c = \sum_{i=1}^{n} a_i s_i$, we define

$$\int_c \omega = \sum_{i=1}^{n} a_i \int_{s_i} \omega$$

Stokes' theorem ensures that

for ω closed $\quad \int_{\partial c} \omega = \int_c d\omega = 0$

and for c a cycle $\quad \int_c d\omega = \int_{\partial c} \omega = 0$

Thus we get a well-defined paring of homology and cohomology

$$\begin{aligned} H_k(M) \times H^k(M) &\longrightarrow \mathbb{R} \\ ([c], [\omega]) &\longmapsto \int_c \omega \end{aligned}$$

THEOREM This pairing give rise to the identities

$$H^k(M) = \mathrm{Hom}_{\mathbb{Z}}(H_k(M), \mathbb{R})$$

$$H_k(M) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(H^k(M), \mathbb{R})$$