„SATZ VOM STETIG GEKÄMMTEN IGEL"

THEOREM On S^n there is a nowhere vanishing vector-field iff n is odd.

PROOF Assume there is a nowhere vanishing vector field X. Then consider the vectorfield $Y(x) = \frac{X(x)}{\|X(x)\|}$. This gives rise to the map $F: S^n \longrightarrow S^n$

$$F(x,t) = \cos(\pi t)x + \sin(\pi t)Y(x)$$

which is a homotopy with $id_{S^n} \simeq_F a$ where a is the antipodal map on S^n. a can be represented by $(n+1)$ reflections. Thus $\deg(a) = (-1)^{n+1}$.
Since $1 = \deg(id_{S^n}) = \deg(a) = (-1)^{n+1}$, for homotopic maps, this homotopy only exist for n odd.
Thus there can't be a nowhere vanishing vectorfield on S^n for n even, and, in fact, $[x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n]$ is a nowhere vanishing vector field on S^n for n odd. $\square$

COROLLARY $\mathbb{R}P^n$ is orientable iff n is odd.

PROOF $\mathbb{R}P^n$ is orientable $\Leftrightarrow$ $\exists$ a nowhere vanishing n-form ω on $\mathbb{R}P^n$
$\Leftrightarrow$ $\exists$ a nowhere vanishing n-form $\pi^*\omega$ on S^n
$$\omega = \sum_{i=1}^{n} a_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+1}$$
$\Leftrightarrow$ $X = \sum \omega(\partial x_i)\partial x_i$ is a no-where vanishing vector field on S^n. $\square$