Notes on Notes on Quantum Field Theory

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1 INTRODUCTION

WARNING: INCOMPLETE AND UNEDITED!

These are notes, remarks and questions which came up while we tried to study the notes of Borcherd's course on quantum field theory [1]. As these were working notes, they may be incomplete, wrong and most likely not helpful.

2 LAGRANGIANS

DEFINITIONS To begin with, an abstract definition of the set of *Lagrangians* is given as

$$\mathcal{V} := \mathrm{Sym}^{\bullet}(\mathbf{R}[\partial_1 \dots \partial_n] \otimes \Phi) \tag{A}$$

where n is the dimension of the *space-time* $\mathbf{R}^n$, ∂_μ are its translation invariant vector fields and Φ is the *space of abstract fields*, a finite dimensional complex vector space with complex conjugation $\star$. This gives us a clean definition and a good start to dig into the theory. Also, it enables us to speak about quantum field theory in a way that physicists won't recognise it. In particular, physicists like to think about *fields* in a more concrete way as functions on space-time, i.e. the space of fields is huge whereas in this approach and the examples given Φ is a finite-dimensional space.

REPRESENTATIONS However, the notion of an *abstract field* and that of a *field* in the physicists' sense are related by what is introduced as a *representation of abstract fields*, the extension to the space of Lagrangians of a map

$$\varrho : \Phi \longrightarrow \mathrm{Map}(\mathbf{R}^n, \mathbf{C})$$

that preserves complex conjugation. We got the impression that the space of such representations will be quite large and possibly studying it could be important. Not much is said about the space of representations in the notes and it seems like this could be a way to not have to deal with a large space of fields right away but start with the small and well-understood space Φ instead and see how much we can find out about Lagrangians from that point of view while the full complexity of the situation is hidden in the space of representations.

Hence the space of representations remains a somewhat open question and some additional remarks regarding its role and properties would be quite helpful.

EXAMPLES Next, examples are given which provide a good opportunity for the non-physicist to actually perform computations in the new framework. The examples seem to be standard physics and implicitly use variational calculus to derive the *Euler-Lagrange Equations* for a Lagrangian. More background on this is given in, say, [2, §1.1].

In [2] the variation of a field φ is called ξ, whereas in this text it is denoted $\delta\varphi$, one of the many meanings of δ we are going to see. Furthermore in this section there is no definition given to the expression δL for a Lagrangian L but only for $\delta \int L$. In particular, δL is (as we will see later, for this expression has no meaning at this stage) not the Euler-Lagrange equation for L, however they are related. The motto for this relation is

$$\delta \int L = \langle \mathrm{Euler-Lagrange}, \delta\varphi \rangle + \mathrm{somethinrelatedtotheconservedcurrentof} L.$$

Also note that making sense of the whole variational business requires us to think of the fields as fields in the traditional sense rather than the formal notion defined as abstract fields above. To stay within the scope of abstract fields, we thought the variation of fields could be shifted to a variation of representation. However nothing along these lines is mentioned or pursued in the text which seems to implicitly make the transition to the traditional notion of fields.

UPPER AND LOWER INDICES Example 2 is the first time we encounter expressions including $\partial_\mu \partial^\mu \varphi$. These include the symbol ∂^μ which is yet to be defined but will not be in the text. So what is ∂^μ?

Thinking that this is only done for notational convenience seems too naïve. Instead there should be a relation $\partial^\mu = g^{\mu\nu}\partial_\nu$ using a the inverse of a metric $g_{\mu\nu}$ to relate the two symbols. While we get away with not worrying about the metric for the time being, this indicates that we have to think about whether this can cause any problems. As soon as we want to allow for non-definite metrics – which physicists do – we will have to spend further thought on this to make sure we get all the signs right in our computations.

From a formal point of view there seem to be several ways to approach the ∂^μ. One of them would be to introduce them as additional symbols in the construction of the space of Lagrangians and then require an additional condition to hold for representations to ensure the ∂^μ fit in appropriately. Another possibility would be to define $\partial^\mu = g^{\mu\nu}\partial_\nu$ but in this case where the $g^{\mu\nu}$ live in our space of Lagrangians as defined in (A). Would they be abstract fields?

THE GENERAL EULER-LAGRANGE EQUATION Finally the generalisation of the results of the previous computations is given for arbitrary Lagrangians. There are a few things we noted about this.

Firstly, the alternating signs we see in equation B can alternatively be made plausible as arising in the following way: Consider L in the graded algebra $\mathbf{R}[t_1, \dots, t_k]$ as a Lagrangian, i.e. there is only one dimension of fields of the form $L \otimes 1$, and its *Euler vector-field*

It's plausible – but this defn. of E-vf seems a bit unusual.

$$eu(L) = -\sum_{j=1}^{k} t_j \frac{\partial}{\partial(t_j \cdot 1)}$$

which has a negative sign as the t_j are of degree 1. Then extend this construction respecting the grading where $\deg \varphi = 0$, $\deg \partial_j \varphi = 1$, $\deg \partial_j \partial_k \varphi = 2$, etc. to get the expression

$$1 \cdot \frac{\partial L}{\partial(1 \cdot \varphi)} - \partial_j \frac{\partial L}{\partial(\partial_j \varphi)} + \partial_j \partial_k \frac{\partial L}{\partial(\partial_j \partial_k \varphi)} - \dots \qquad \text{(B)}$$

as in the text.

Secondly, to think of the Euler-Langrange equations and their derivatives as an ideal of the ring of Langrangians, requires us to evaluate expressions such as $\partial L / \partial \varphi$ or $\partial / \partial(\partial_j \varphi)$, etc. as formal derivatives of L in the variables φ, $\partial(\partial_j \varphi)$ etc.

Thirdly, the statements at the very end of section 2.3 require some additional thought to understand the sense in which "the quotient [of $\mathcal{V}$ by the the ideal of Euler-Langrange equations] is a ring over $\mathbf{C}[\partial_\mu]$".

3 SYMMETRIES AND CURRENTS

This section begins by considering Lagrangians for complex scalar fields, i.e. Lagrangians where the space of abstract fields is $\mathbf{C}^2$ with a basis $\{\varphi, \varphi^\star\}$. It looks like the complex scalar field is used throughout section 3.1, whereas the real scalar field is used in section 3.2. Both sections give examples for finding symmetries given by a Lagrangian. It is not clear, though, why these specific examples were chosen and in which sense the techniques exhibited can be generalised.

OBVIOUS SYMMETRIES The infinitesimal deformation of the fields for a Lagrangian is given by the operator $\delta : \mathcal{V} \longrightarrow \Omega^1(\mathcal{V})$ with the target space being the module with basis $\delta\varphi$. This operator is called *natural* – probably in the colloquial sense. It is required to satisfy a Leibniz condition $\delta(ab) = (\delta a)b + a(\delta b)$ as well as $\delta \partial_\mu = \partial_\mu \delta$.

The notation used here is a bit confusing as the symbol δ is being used in three ways:

- Possibly in the sense of section 2, denoting a variation of the fields. From the text it is not clear what we want this to be formally.
- In the sense of basis elements of for $\Omega^1(\mathcal{V})$ – an approach to formalise the above concept.
- In the sense of the map $\mathcal{V} \to \Omega^1(\mathcal{V})$ just defined, that could link the meanings from the first two points.

It is not really clear which meaning of δ applies at which place in the text. I do suppose, though, that more than one is used and tieing the picture from section 2 together with the formal picture that has just been introduced is the point of all this.

The 'condition' $\delta\partial_\mu = \partial_\mu\delta$ seems to be the key to confusion as its right-hand side* doesn't make sense: No matter in which way we think about δ, we'll have to think of ∂_μ as an element of $\mathcal{V}$ which it isn't – just elements of the form $\partial_\mu \otimes \varphi$ are (and $1 \notin \Phi$ rules out the possibility of ∂_μ implicitly meaning $\partial_\mu \otimes 1$). Assuming that we want the equations to hold – which we definitely want to make the subsequent computations – the best way to make sense of this was assuming that this equation is actually used to define the meaning of expressions of the form $\partial_\mu\delta\cdot$.

It is not entirely clear *why* the notation in the original order won't do but introducing new notation definitely isn't forbidden. To make everything work as desired we'll have to also require that ∂_μ used in this way, is linear and satisfies a Leibniz rule.

Sticking to all these rules and using the notation the 'moral' of the following computations is that for a Lagrangian L we have

$$\delta L = \sum_{j=1}^{k} \delta\varphi_j \cdot EL_j + \sum_{\mu=1}^{n} \partial_\mu J^\mu$$

where $\{\varphi_j\}_{j=1}^k$ is a basis for Φ, EL_j denote Euler Lagrange equations and J^μ is a conserved current. The latter is an expression in $\delta\varphi_j$. Replacing these by the infinitesimal generators of appropriate actions will give a *Noether current* which can neatly be thought of as an $n-1$-form that is determined up to a closed $n-2$-form.

This looks like an element of $H^{n-1}(\cdot)$. But of what?

*The meaning of the expression on the left-hand side isn't obvious either, but thinking about what it should be intuitively, reveals that we should think of ∂_μ as a shorthand notation for the map $Sym^\bullet(m_{\partial_\mu} \otimes \mathrm{id}_\Phi)$ here, where m_{∂_μ} denotes multiplication by ∂_μ in $\mathbf{R}[\partial_1, \ldots, \partial_n]$.

IMAGINARY In the computations on page 15 the infinitesimal generator for $\varphi \mapsto e^{i\vartheta}\varphi$ is seen to be $i\varphi$ (as the Lie algebra of $S^1 \subset \mathbf{C}$ is $i\mathbf{R}$ rather than $\mathbf{R}$). Still, in the computations following this the i is divided out of the expression. Similarly, imaginary units will be dropped later on as well, say in the beginning of section 5. Is there any significance to writing the i? If there isn't, why?

NOT-SO-OBVIOUS SYMMETRIES This section features another symmetry: $\varphi \mapsto e^{\vartheta\partial_\mu\varphi}$. By pretending that ∂_μ is just a number, we derive

$$\frac{\partial}{\partial\vartheta} e^{\vartheta\partial_\mu}\varphi\Big|_{\vartheta=0} = \partial_\mu e^{\vartheta\partial_\mu}\varphi\Big|_{\vartheta=0} = \partial_\mu\varphi.$$

Now ∂_μ isn't a number and a remark on why this goes through would be helpful. Is it another 'formal analogy'? Do we use an expansion for exp?

Towards the end of the section there are two more things that raise questions which don't seem essential for what follows but may improve general understanding: Firstly the bit "[...] the above transformation is a symmetry of the physics described by the Lagrangian". This is the first time physics come into play – so for the non-physicists, elaborating this a little might help. Secondly, at the very end of the section an index is raised explicitly and a metric is being used. As noted before on page 2, the significance of this is not quite clear.

THE ELECTROMAGNETIC FIELD The Lagrangian for the electromagnetic field is introduced in the language that has been established and it is shown how the results are related to the traditional Maxwell equations (which your favourite physicist is keen on explaining to you). The footnote on page 18 mentions another cool way to think about the Maxwell equations in terms of the curvature form of a $U(1)$-bundle which isn't important here, though.

elaborate this?

HOMOLOGICAL ALGEBRA Finally many ideas of the section are summed up in a more axiomatic approach, using universal derivation modules to define the objects $\Omega^j(V)$ as well as δ. This approach seems to avoid the troubles and ambiguities we found in the section on obvious symmetries (cf. page 3). We arrive at having a double complex $E_{jk} = \bigwedge^{n-j}(\mathbf{R}^n) \otimes \Omega^k(V)$ with horizontal differential $\mathrm{id} \otimes \delta$ induced by the differential δon $\Omega^\bullet(V)$ and vertical differential ∂ that is given by $\partial(\omega \otimes v) = (dx^\mu \wedge \omega) \otimes \partial_\mu v$, formally resembling the differential of the de Rham complex.

In fact, when taking the base ring R in the construction to be $C^\infty(M)$ and V to be $C^\infty(M)$ as well, as a module over itself, along with the usual derivation $\delta : C^\infty(M) \to C^\infty(M)$ to induce the horizontal differential, we have ∂ as the differential on the vertical subcomplex $E_{\bullet k} = \bigwedge^\bullet(\mathbf{R}^n) \otimes C^\infty(M) = \Omega^\bullet_{dR}(M)$ turn out to be exactly the de Rham differential.

This is an elegant approach that isn't taken up again later in the notes. What is its significance? What can the homology groups of the double complex tell us? discuss with Victor

4 FEYNMAN PATH INTEGRALS

In this section, the *Feynman path integrals* for a Lagrangian L is introduced as

$$I[J] = \int \exp\left(i \int L[\varphi] + J\varphi \; d^n x\right) D\varphi.$$

This notation omits the set over which we integrate. Having in mind the discussion that follows this definition, though, it is clear that we don't integrate over the space of abstract fields Φ as defined at the beginning of the paper, but rather over the infinite dimensional 'physical' space of fields $C^\infty(\mathbf{R}^n)$, which is what makes the integral troublesome in the first place.

Then a heuristic explanation of the terms appearing in the Lagrangian

$$\partial_\mu \varphi \partial^\mu \varphi - m^2/2\varphi^2 + \lambda/4!\varphi^4 \tag{C}$$

is given. This Lagrangian, its parts and its Feynman integral are studied in this section. The main idea in doing so is to evaluate and study a similar finite dimensional integral first and then define the actual Feynman integral to be an infinite dimensional formal analogue of the results for the computations we were able to make.

THE FREE FIELD CASE This generalisation is done at first for the *free field*, meaning that there is no φ^4 term present – according to the heuristics given in the beginning of the section it is the φ^4 terms that lead to interesting interactions, hence not having them makes the field 'free'. The first step of the manipulation of $I[J]$,

$$\begin{aligned} I[J] &= \int \exp\left(i \int \partial_\mu \varphi \partial^\mu \varphi + R \; d^n x\right) D\varphi \\ &= \int \exp\left(i \int \partial_\mu(\varphi \partial^\mu \varphi) - \varphi \partial_\mu \partial^\mu \varphi + R \; d^n x\right) D\varphi \\ &= \int \exp\left(i \int -\varphi \partial_\mu \partial^\mu \varphi + R \; d^n x\right) D\varphi \end{aligned}$$

uses integration by parts and seems to use some rapid decay condition that isn't discussed further to get rid of the $\partial_\mu(\varphi\partial^\mu\varphi)$ part. The outcome of the formal manipulations contains an undefined factor of $\pi^\infty/\sqrt{\det(A)/i}$ which is equal to $I[0]$, the so-called *vacuum state*. Hence, we consider the ratio $I[J]/I[0]$ to formally get rid of the undefined expressions. This leaves us with just one more undefined bit in the result which is made sense of using distributions and *fundamental solutions* Δ, yielding the definition

$$\frac{I[J]}{I[0]} := \exp\left(\frac{-i}{4}\iint J(x_1)\Delta(x_1-x_2)J(x_2)\ d^nx_1d^nx_2\right) \tag{D}$$

FREE FIELD GREEN'S FUNCTIONS The next step is to do the power series expansion for the above expression and manipulating its terms to give

$$\frac{I[J]}{I[0]} = \sum_{j=0}^{\infty}\frac{(-i)^j}{4^j j!}\int\cdots\int\prod_{k=1}^{j} J(x_{2k-1})J(x_{2k})\Delta(x_{2k-1}-x_{2k})\ d^nx_1\ldots d^nx_{2j}.$$

Next, we take all *pairs* of variables in $x_1,\ldots,x_j$ and note that replacing the pair occuring in each factor of the product in the above expression doesn't change the integral. As we will see in the next section, there are $(2j)!/j!2^j$ distinct pairs. Furthermore, taking the sum over all distinct pairs will let us replace the products of Δ by *Green's functions* as defined in the text:

$$\sum_{p\ \text{pairing}}\prod_{k=1}^{j}\Delta(p[x_{2k-1}]-p[x_{2k}]) = G_{2j}(x_1,\ldots,x_{2j}).$$

Substituting this into the previous equation yields

$$\frac{I[J]}{I[0]} = \sum_{j=0}^{\infty}\frac{(-i)^j}{2^j(2j)!}\int\cdots\int J(x_1)\cdots J(x_{2j})G_{2j}(x_1,\ldots,x_{2j})\ d^nx_1\ldots d^nx_{2j}.$$

This is pretty close to the formula given in the text, but not quite it. It doesn't seem obvious how to get rid of the factor $(-1/2)^j$ in each summand graciously. However, the notation in the text suggests that handling those is important as the sum given there doesn't use any odd values for j, i.e. the N given as variable there corresponds to $2j$ in the notation used above.

can we resolve this problem?

The technique of looking at pairs of points is a technique that is used in the construction of *Feynman diagrams* in the next section.

THE NON-FREE CASE The next step is to introduce the φ^4-term that "leads to interesting interactions" physically, thus considering the full Lagrangian as defined in equation C. As in the previous paragraph, the power series expansion for the exponential is used, leading us to a computation where, with L being the free field Lagrangian,

$$\int \exp\left[i\int Ld^nx\cdot\sum_j\frac{i^j}{j!}\left(\int J\varphi+\frac{\lambda}{4!}d^nx\right)^j\right]D\varphi$$

is said to yield and expressions with terms of the form

$$\int \exp\left[i\int Ld^nx\cdot\left(\int\varphi(x_1)^{n_1}J(x_1)d^nx_1\right)\cdots\left(\int\varphi(x_N)^{n_N}J(x_N)d^nx_N\right)\right]D\varphi.$$

While we're not caring about constants here, it doesn't seem obvious how we arrive at these terms. Looking at the power of the integrals in the upper line and expanding it suggests that there should be mixed variables $x_j$in the resulting integral(s), say terms containing factors $J(x_1)J(x_2)$. However, the products in the given results only use one of the variables per integral. How is this separation achieved?

In the final bit of the section we're told that similarly to the free case this leads to Green's functions – with singularities in this case. How exactly does the construction analogous to that in the finite dimensional case work? Do we simply replace J by $J + \varphi^3$ in the computations for the free field case?

5 0 DIMENSIONAL QFT

The first example is given, in dimension 0. Physicists we talked to laughed at the uselessness of this "trivial" example. Yet, it turned out there's more than enough mathematics going on here to make the investigation worthwhile. Also, this is most likely the only case we can properly compute and understand using basic methods, as – under the assumption that $\varphi = \varphi^\star$ in the notation of the first section – integrating over all fields is just integrating over the real line.

Next, we define $Z[\lambda, j] = \int LD\varphi$ to be the Feynman path integral for our Lagrangian $L = -1/2\varphi^2 - \lambda/4!\varphi^4$ and it is "easy" to see that this is reasonably nice everywhere but at $\lambda = 0$. Thus we ignore the latter fact and to an expansion at $\lambda = 0$. This works formally, but how can we justify doing it? What does the singularity at $\lambda = 0$ mean for us? We have also once more "ignored" a factor i.

Doing all the related computations and combining them, we end up with an expression containing the factor

$$\frac{(4m+2k)!}{(2m+k)!2^{2m+k}} \cdot \frac{1}{(4!)^m m!(2k)!} \cdot (-\lambda)^m \cdot j^{2k} \qquad \text{(E)}$$

The factors in the first two fractions have following combinatorial meaning, starting off with a set of $4m + 2k$ points

$(4m + 2k)!$ The number of permutations of $4m + 2k$ points.

$(2m + k)!$ Group the $4m + 2k$ points into pairs. This is the number of permutations of complete pairs.

2^{2m+k} Given on of these pairs, there are 2 permutations of its points. Given $2m + k$ points, this is the number of permutations *within* the pairs.

$\frac{(4m+2k)!}{(2m+k)!2^{2m+k}}$ Piecing the previous three together this is the number of distinct ways to split up a set of $4m + 2k$ elements into pairs.

$(4!)^m$ Wanting to designate m sets of 4 points each within our original set, this is the number of permutations within these m sets.

$m!$ Given the m sets of 4 points, this is the number of permutations of the whole sets.

$(2k)!$ Having already "used" $4m$ points in the previous two steps, there are $2k$ points left. This is the number of permutations of the remaining points.

This listing may give additional clarification to what is already explained in the text about how the formulæ relate to diagrams with valence 1 and valence 4 vertices. It can't replace drawing a few diagrams and doing the associated computations, though. The fact that we're dealing with permutations and permutations groups here, suggests that everything can be formalised in terms of symmetric groups. This can be done and I'll repat the argument given in the text in this more formal setting:

As seen in the listing of coefficients above, we are doing two things here. The first is pairing points, which can be done by considering all possible permutations and identifying all those that don't cross the boundaries of pairs. In terms of groups this is formalised by the *wreath product* (see [**?**] for more details) of the two symmetric groups involved, giving the group

find reference

$$H_1 = S_2 \wr S_{2m+k} \subset S_{4m+2k} = \Sigma.$$

Secondly, we want to form m quadruples, giving rise to the same construction and permute the $2k$ remaining points, giving rise to the subgroup

$$H_2 = (S_4 \wr S_m) \times S_{2k} \subset \Sigma.$$

Both H_1 and H_2 act on S_{4m+2k}. Using the H_1-action we get Σ/H_1 of which we consider the H_2-orbits. Using $[x]$ and $[[x]]$ to denote H_1-cosets and double cosets respectively, we get

Afterthought: Is all of this sound? How do the group actions work? Is the action of H_2 on Σ/H_1 well-defined?

$$\frac{|\Sigma|}{|H_1|} = \sum_{[[x]]\in H_2\backslash\Sigma/H_1} |H_2[x]| = \sum_{[[x]]\in H_2\backslash\Sigma/H_1} \frac{|H_2|}{\left|\mathrm{Stab}_{[x]} H_2\right|}$$

where $[x]$ is stabilised by H_1^x, the x-conjugate of H_1, intersected with H_2 of course. Note, that by the discussion of what H_1 and H_2 do above, the set of double cosets $H_2\backslash\Sigma/H_1$ corresponds exactly to the (reduced) diagrams and the set $H_1^x \cap H_2$ corresponds to the automorphisms leaving the graph represented by x invariant. Thus we get

this seems to work for a right action of H_1 only. what's the importance of that?

$$\frac{|\Sigma|}{|H_1|\,|H_2|} = \sum_{[[x]]\in H_2\backslash\Sigma/H_1} \frac{1}{|H_1^x \cap H_2|} = \sum_{g \text{ graphs}} \frac{1}{|\mathrm{Aut}(g)|}$$

as seen in the text.

Next, the first terms of the sum are computed to give an example and a few paragraphs are dedicated to dealing with the fact that we weren't allowed to do the the expansion we did in the first place. Also, *Borel summation* is mentioned as a method to help computing the resulting sums. The treatment of both topics is quite brief and the latter ends with the remark that there is a problem with Borel summation as "it does not pick up non-perturbative effects".

OTHER GRAPH SUMS Now that we started dealing with graphs and writing down weighted sums of them, we have our first go at summing them differently – seeing that $\log Z[\lambda, j]$ is nothing but the sum over connected graphs only. In the computation of the coefficients, note that the automorphism group of n disjoing copies of a graph is again the wreath product $\mathrm{Aut}(G) \wr S_n$, giving the equality $|\mathrm{Aut}(nG)| = |Aut(G)|^n n!$ used in the text.

drawing of vacuum bubble?

Also, it is pointed out that dividing by the Feynman path integral for not field $Z[\lambda, 0]$ corresponds to only summing diagrams with univalent vertices: It is precisely the univalent vertices that have a factor of j attached to them. Thus there can't be any for $j = 0$. It is mentioned that this process is referred to as removing the *vacuum bubbles.* The term seems to be coined by the fact that physicists refer to $Z[\lambda, 0]$ as the *vacuum state* and the corresponding diagrams look like bubbles. As a vacuum bubble is a diagram whose coefficient is constant in j, another way to get rid of them is to apply d/dj, which will be done in the next section.

THE CLASSICAL FIELD Trying to also appoximate the classical theory, a naïve approach would be to "average" the over all fields:

$$\varphi_n = \frac{\int \varphi \exp L(\varphi) D\varphi}{\int \exp L(\varphi) D\varphi} = \frac{d}{dj} \log Z[\lambda, j] = j - \frac{\lambda j^3}{3!} - \frac{\lambda j}{2} + O(\lambda^2)$$

This quickly turns out to be the wrong approach as this fied doesn't satisfy the Euler-Lagrange equation

$$\begin{aligned} EL(\varphi_n) &= \frac{dL}{d\varphi}(\varphi_n) = -\varphi_n - \frac{\lambda}{4!}\varphi_n^3 + j \\ &= -j + \frac{\lambda j^3}{3!} + \frac{\lambda j}{2} - \frac{\lambda}{3!}\left(j - \frac{\lambda j^3}{3!} - \frac{\lambda j}{2}\right) + j + O(\lambda^2) \\ &= \text{linear in } \lambda + O(\lambda^2) \neq 0 \end{aligned}$$

The right approach to this is to introduce a new variable $\hbar$ and use use the method of stationary phase, giving a field that automatically satisfies the Euler-Langrange equations:

more details on stationary phase? Ask Victor to explain details again

$$\varphi_{cl}(\lambda, j, \hbar) = \hbar \frac{d}{dj} \log \int \exp\left[\frac{1}{\sqrt{\hbar}}\left(-\frac{1}{2}\varphi - \frac{\lambda}{4!}\hbar^{3/2}\varphi^4 + j\varphi\right)\right] D\varphi$$

A combinatorical approach to this is given in the text.

THE EFFECTIVE ACTION The effective action is defined and discussed in this section. It seems to be motivated by the aim to achieve greater analogy with the classical theory than φ_{cl} as constructed previously gives. As peeking into the literature, say [3, Chapter 11], suggests, the background for this analogy lies further in the world of physics.

What the effective action seems to do is take a field and give us a Lagrangian for that field that isn't exactly the Lagrangian we started with but still related. Using *1-PI graphs* the combinatorical aspect of its construction is thorougly discussed and computations are made using the techniques of summing graphs with attached coefficients as before.

If I am not mistaken, the factor attached to the bottom-right graph on at the top of page 32 should be $\lambda^2\varphi^4/16$ rather than $\lambda\varphi^3/8$ as we have four two-fold symmetries: three switching around the edges and one interchanging the vertices.

REFERENCES

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[3] Michael E. Peskin and Daviel V. Schroeder. *An Introduction to Quantum Field Theory.* Perseus Books, 1995.