POINCARÉ LEMMA

LEMMA $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ $(x,t) \mapsto x$, $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ $x \mapsto (x,0)$
Then the ~~maps~~ induced maps on cohomology $\pi^* = H^\bullet(\pi)$ and $s^* = H^\bullet(s)$ are mutually inverse isomorphisms.

PROOF Clearly $\pi \circ s = id_{\mathbb{R}^n}$ and thus $(\pi \circ s)^* = id_{H^\bullet(\mathbb{R}^n)}$.
It remains to show that $(s \circ \pi)^* = \pi^* \circ s^* = id_{H^\bullet(\mathbb{R}^n \times \mathbb{R})}$
To do this, we construct a chain homotopy between the maps of complexes $\pi^* \circ s^* : \Omega^\bullet(\mathbb{R}^n \times \mathbb{R}) \to \Omega^\bullet(\mathbb{R}^n \times \mathbb{R})$ and $id_{\Omega^\bullet(\mathbb{R}^n \times \mathbb{R})}$, i.e. find a „chain" map of degree -1 st

$$\cancel{id} - \pi^* \circ s^* = \pm(Kd \pm dK)$$

Define K by

$$\omega \longmapsto \begin{cases} 0 & \text{if } \omega = \pi^*(\phi) f(x,t) \\ \pi^*(\phi) \int_0^t f(x,u)\,du & \text{if } \omega = \pi^*(\phi) f(x,t) \wedge dt \end{cases}$$

It can be checked that this is indeed a chain homotopy. Thus we have for a closed form ω:

$$\begin{aligned}(id - \pi^* \circ s^*)(\omega) &= \pm(Kd \pm dK)(\omega) \\ &= \pm d(K\omega)\end{aligned}$$

which is ~~an~~ exact form and thus 0 on cohomology.
It follows that $id(\omega) = \pi^* \circ s^*(\omega)$ on cohomology.

$\square$

LEMMA (POINCARÉ) $H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$

PROOF This follows by knowledge of $H^k(\mathbb{R})$ and induction on k using the previous lemma.

$\square$

COROLLARY $H^k(M) \cong H^k(M \times \mathbb{R})$

PROOF Follows from the Poincaré-lemma when working with charts on $M \times \mathbb{R}$ of the form $\mathcal{U} \times \mathbb{R}$ where $\mathcal{U}$ belongs to a chart on M.

$\square$

COROLLARY $f \simeq g \Rightarrow f^* = g^*$

PROOF With a homotopy F, $f^* = s_1^* \circ F^* = s_0^* \circ F^* = g^*$. $\square$

! We can do similar things for compactly supported cohomology.

LEMMA The maps π_* and e_* as defined below are mutually inverse isomorphisms.

PROOF $\pi : M \times \mathbb{R} \to M$ $\quad \rho : M \times \mathbb{R} \to \mathbb{R}$ $\quad e \in \Omega_c^1(\mathbb{R})$ st $\int_{\mathbb{R}} e = 1$

then define

$$e_* : \Omega_c^{k-1}(M) \to \Omega_c^k(M \times \mathbb{R}) \qquad \omega \longmapsto \pi^*\omega - \rho^* e$$

This map descends to a map on H_c^* where it is an isomorphism with cohomological inverse induced by

$$\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M) \qquad \omega \longmapsto \begin{cases} \pi^*(\varphi) f(x,t) & : \quad 0 \\ \pi^*(\varphi) f(x,t) \wedge dt & : \quad \varphi \int_{-\infty}^{\infty} f(x,t)\,dt \end{cases}$$

We can show that $d\pi_* = \pi_* d$ and we have a chain homotopy

$$K : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^{\bullet-1}(M) \qquad \omega \longmapsto \begin{cases} \pi^*(\varphi) f(x,t) & : \quad 0 \\ \pi(\varphi) f(x,t) \wedge dt & : \quad \pi^*\varphi\left(\int_{-\infty}^{t} f(x,u)\,du - \int_{-\infty}^{t} e \cdot \int_{-\infty}^{\infty} f(x,u)\,du\right) \end{cases}$$

The rest follows as before

$\square$

LEMMA! (POINCARÉ) $H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq 0 \end{cases}$