Lie $U(n) \otimes \mathbf{C} \simeq \mathrm{End}(n, \mathbf{C})$

We claim that the complexified Lie algebra of the unitary group in n dimensions is isomorphic to the endomorphism group of an n-dimensional vector space. To show this, we begin be recalling some fact about the unitary group, then we spell out what the Lie algebras in question are and finally we give the Lie algebra isomorphism.

The unitary group $U(n)$ is the unitary group of an n-dimensional vector space with a hermitian inner product. Its element are the endomorphisms A satisfying $A^*A = \mathrm{Id}$. Differentiating this equation gives that the Lie algebra $\mathbf{u}(n)$ of $U(n)$ consists of endomorphisms satisfying $A^* + A = 0$, i.e. those which are skew-hermitian. The Lie bracket on $\mathbf{u}(n)$ is the usual one: $[A, B] = AB - BA$.

By choosing a basis for V and working with matrices, it becomes apparent that $\dim_{\mathbf{R}} \mathbf{u}(n) = n$, as an anti-hermitian $n \times n$-matrix is exactly determined by the $n(n-1)/2$ *complex* entries above the diagonal and n *real* entries on the diagonal.

Also note that, in analogy to the anti-hermitian morphisms of $\mathbf{u}(n)$, we can define the space of hermitian morphisms $\mathbf{u}(n)^\perp = \{A | A^* - A = 0\} \subset \mathrm{GL}(n, \mathbf{C})$. As non-zero morphisms cannot be both hermitian and anti-hermititan, we get an $\mathbf{R}$-vector space isomorphism $\mathrm{End}(n, \mathbf{C}) \simeq \mathbf{u}(n) \oplus \mathbf{u}(n)^\perp$ for dimensional reasons. Thus we can write any endomorphisms uniquely as a sum of an anti-hermitian and a hermitian one. Finally, note that multiplication be the scalar i gives an isomorphism $\mathbf{u}(n) \simeq \mathbf{u}(n)^\perp$.

The Lie algebras Complexification gives $\mathbf{u}(n)^{\mathbf{C}} = \mathbf{u}(n) \otimes_{\mathbf{R}} \mathbf{C}$. Analogously to writing complex numbers as 2×2-matrixes of reals, we write elements of $\mathbf{u}(n)^{\mathbf{C}}$ as 2×2-matrices with entries in $\mathbf{u}(n)$ having the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ where A and B are elements of $\mathbf{u}(n)$. Alternatively, this is an element of $\mathrm{End}(2n, \mathbf{C})$. Using the Lie bracket from $\mathrm{End}(2n, \mathbf{C})$, we compute

$$\begin{aligned} &\left[\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \begin{pmatrix} A' & -B' \\ B' & A' \end{pmatrix} \right] \\ = \ &\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} A' & -B' \\ B' & A' \end{pmatrix} - \begin{pmatrix} A' & -B' \\ B' & A' \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \end{aligned}$$

$$= \left(\begin{array}{cc} AA' - BB' - A'A + B'B & -AB' - BA' + A'B + B'A \\ BA' + AB' - B'A - A'B & -BA' + AA' + B'B - A'A \end{array} \right)$$

and thus see that this bracket restricts to the subset $\mathbf{u}(n)^{\mathbf{C}} \subset \mathrm{End}(2n, \mathbf{C})$.

The other Lie algebra we are dealing with, $\mathrm{End}(n, \mathbf{C})$, has the standard bracket, not requiring further discussion.

The Isomorphism Piecing together all of the above, we can write down the linear map

$$\begin{array}{rccl} \varphi : & \mathbf{u}(n)^{\mathbf{C}} & \longrightarrow & \mathrm{End}(n, \mathbf{C}) \simeq \mathbf{u}(n) \oplus \mathbf{u}(n)^{\perp} \\ & \left(\begin{array}{cc} A & -B \\ B & A \end{array} \right) & \longmapsto & A + iB. \end{array}$$

As both A and B are from $\mathbf{u}(n)$ and as iB is in $\mathbf{u}(n)^{\perp}$, φ is injective, thus bijective. It is also compatible with the Lie brackets,

$$\begin{array}{cl} & \left[\varphi \left(\begin{array}{cc} A & -B \\ B & A \end{array} \right), \varphi \left(\begin{array}{cc} A' & -B' \\ B' & A' \end{array} \right) \right] \\ = & [A + iB, A' + iB'] \\ = & AA' + iAB' + iBA' - BB' - A'A - iA'B - iB'A + B'B \\ = & (AA' - BB' - A'A + B'B) + i(AB' + BA' - A'B - B'A) \\ = & \varphi \left(\begin{array}{cc} AA' - BB' - A'A + B'B & -AB' - BA' + A'B + B'A \\ BA' + AB' - B'A - A'B & -BA' + AA' + B'B - A'A \end{array} \right) \\ = & \varphi \left(\left[\left(\begin{array}{cc} A & -B \\ B & A \end{array} \right), \left(\begin{array}{cc} A' & -B' \\ B' & A' \end{array} \right) \right] \right), \end{array}$$

making φ a Lie algebra isomorphism as required.