THE LEIBNIZ RULE ON PRINCIPAL BUNDLES

ABSTRACT. When differentiating maps involving the group action on a principal bundle, the Leibniz rule needs to be used in some cases. The reason for this is frequently hidden in notation which suppresses the product causing this. It is made explicit here.

Let G be a Lie group with multiplication map $m : G \times G \to G$. G acts on itself by left multiplication λ, right multiplication ϱ and conjugation Ad. Given a manifold B, a *principal G-bundle on B* is a locally trivial fibre bundle $\pi : P \to B$ with a right action of G on P that is free and transitive.

The right action of G can be represented with various degrees of generality. Most generally it is a a map

$$R : P \times G \longrightarrow P$$

which is compatible with the multiplication and unit in G, that is, the following two diagrams commute:

(1)
$$\begin{array}{ccc} P \times G \times G & \xrightarrow{R\times\mathrm{id}} & P \times G \\ {\scriptstyle \mathrm{id}\times m}\downarrow & & \downarrow{\scriptstyle R} \\ P \times G & \xrightarrow[R]{} & P \end{array} \qquad\qquad \begin{array}{ccc} P & \xrightarrow{\mathrm{id}\times e} & P \times G \\ & {\scriptstyle \mathrm{id}_P}\searrow & \downarrow{\scriptstyle R} \\ & & P \end{array}$$

For a fixed point $p \in P$ the action gives rise to a map

$$R^p : G \longrightarrow P \qquad g \longmapsto R(p, g)$$

and, likewise, for a fixed $g \in G$ it gives rise to an invertible map

$$R_g : P \longrightarrow P \qquad p \longmapsto R(p, g).$$

Using the last map, we can re-write the condition of the diagrams (1) commuting as $R_h \circ R_g = R_{gh}$ and $R_e = \mathrm{id}_P$ for all $g, h \in G$.

P being a *locally trivial* fibre bundle with a free, transitive group action spells out as follows: For sufficiently fine open covers $\{U_\alpha\}_{\alpha\in A}$ of B, there are trivialisations for preimages $P_\alpha = \pi^{-1}(U_\alpha)$ of the open sets in the cover:

$$\varphi_\alpha : P_\alpha \xrightarrow{\sim} U_\alpha \times G$$

On the target space of the trivialisation map φ_α we have projection maps p_1 and p_2 to the first and second factor. p_1 is compatible with the bundle's projection map π: $p_1 \circ \varphi_\alpha = \pi_{|P_\alpha}$. Furthermore we can define the *unit section* $\sigma_\alpha : U_\alpha \to P_\alpha$ by requiring that $p_2 \circ \varphi_\alpha \circ \sigma_\alpha(b) = e$ for all $b \in B$.

Putting all of the above together gives the following diagram:

$$\begin{array}{ccccccc} P & \supset & P_\alpha & \xrightarrow[\sim]{\varphi_\alpha} & U_\alpha \times G & \xrightarrow{p_2} & G \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_{|P_\alpha}}\downarrow\uparrow{\scriptstyle \sigma_\alpha} & {\scriptstyle p_1}\swarrow & & & \\ M & \supset & U_\alpha & & & & \end{array}$$

The setup of these maps ensures that for each point $p \in P_\alpha$ the action of the group element associated to it via the trivialisation moves the value of the unit section at the corresponding basepoint to p itself:

$$R_{p_2 \circ \varphi_\alpha(p)} \sigma_\alpha(\pi(p)) = p$$

We want to think about this equation in terms of an identity map. As the point p is used both as the element given to the map *and* as a parameter determining the exact map $R_{p_2 \circ \varphi_\alpha(p)}$ used, additional notation is needed to do this. Write the map $R^\alpha = \mathrm{id}_{P_\alpha}$ as

$$P_\alpha \xrightarrow{\Delta} P_\alpha \times P_\alpha \xrightarrow{\pi \times \varphi_\alpha} U_\alpha \times (U_\alpha \times G) \xrightarrow{\sigma_\alpha \times p_2} P_\alpha \times G \xrightarrow{R} P_\alpha .$$

Now the product implicitly used in the equation above is apparent and it is clear that the Leibniz rule has to be used for differentiation. Differentiating at $p \in P_\alpha$ yields:

$$\begin{aligned}
\mathrm{id}_{T_p P_\alpha} &= D_p R^\alpha \\
&= D_{(\sigma_\alpha \pi(p),\, p_2 \varphi_\alpha(p))} R \, D_{(\pi(p),\, \varphi_\alpha(p))} (\sigma_\alpha \times p_2) \, D_{(p,\, p)} (\pi \times \varphi_\alpha) \, D_p \Delta \\
&= D_{(\sigma_\alpha \pi(p),\, p_2 \varphi_\alpha(p))} R \left(D_{\pi(p)} \sigma_\alpha \, D_p \pi ,\, D_{\varphi_\alpha(p)} p_2 \, D_p \varphi_\alpha \right) \\
&= D_{\sigma \pi(p)} R_{p_2 \varphi_\alpha(p)} \, D_{\pi(p)} \sigma_\alpha \, D_p \pi + D_{p_2 \varphi_\alpha(p)} R^{\sigma_\alpha \pi(p)} \, D_{\varphi_\alpha(p)} p_2 \, D_p \varphi_\alpha
\end{aligned}$$

Unfortunately, this level of detail is rather cumbersome. Thus, after convincing oneself of this fact, it seems advisable to drop as much of the notation used above as is feasible in the current context and to revert to the shorthands that are commonly found in the literature.

Taking this to extremes, we can omit the explicit statement of the basepoint for differentiation, denote $p_2 \circ \varphi_\alpha(p) =: g$, omit all the multiplication maps introduced earlier and denote right multiplication with g by adding $.g$ at the right of the expression. As a result, the monstrous equation above can be written as

$$\mathrm{id} = D\sigma_\alpha \, D\pi \, .g + \varphi_\alpha . Dg \; .$$

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