Categories Recall that a small category $\mathcal{C}$ consists of sets of objects $Ob_{\mathcal{C}}$ and morphisms $\mathrm{Mor}_{\mathcal{C}}$ such that there is a morphism id_A for every object A, every morphism φ has a source object $s(\varphi)$ and a target object $t(\varphi)$, morphisms φ and ψ can be composed if $t(\psi) = s(\varphi)$ where this composition is associative:

$$\begin{tikzcd}\mathrm{Mor}_{\mathcal{C}}\ {}_t\!\times_s \mathrm{Mor}_{\mathcal{C}} \arrow[r, "\circ"] & \mathrm{Mor}_{\mathcal{C}} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{Ob}_{\mathcal{C}} \arrow[r, "\mathrm{id}"] & \mathrm{Mor}_{\mathcal{C}}\end{tikzcd}$$

We say that φ is as morphism from an object A to an object B, if $s(\varphi) = A$ and $t(\varphi) = B$ and denote the set of all such morphisms as $\mathrm{Mor}_{\mathcal{C}}(A, B)$.

Definition A *groupoid* is defined to be a category in which every morphism is an isomorphism. That is, we have a map $i : \mathrm{Mor}_{\mathcal{C}} \to \mathrm{Mor}_{\mathcal{C}}$ that assigns to every morphism its inverse. Thus we can extend the diagram above to be:

$$\begin{tikzcd}\mathrm{Mor}_{\mathcal{C}}\ {}_t\!\times_s \mathrm{Mor}_{\mathcal{C}} \arrow[r, "\circ"] & \mathrm{Mor}_{\mathcal{C}} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{Ob}_{\mathcal{C}} \arrow[r, "\mathrm{id}"] & \mathrm{Mor}_{\mathcal{C}} \arrow[r, "i"] & \mathrm{Mor}_{\mathcal{C}}\end{tikzcd}$$

If $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *is* non-empty for any two objects, A and B, the groupoid is said to be *connected* or *transitive*. In this case all objects and their automorphism groups are isomorphic. Sometimes the objects of $\mathcal{C}$ are referred to as *vertices* and their automorphism groups as *vertex groups*.

Relation to groups The most basic example for a groupoid is a group. Given a group G, we can define a groupoid $\mathcal{G}$ with a single object $*$ and $\mathrm{Mor}_{\mathcal{G}} = G$ where composition of maps is given by the group's multiplication. In fact, every connected groupoid is equivalent as a category to a groupoid of this form.

Proof: Let $\mathcal{C}$ be a connected groupoid with automorphism group G; let $\mathcal{G}$ be the groupoid for G. Every set of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is in bijection with the automorphisms of any object by virtue of composition of morphisms. Now, choose

1. a designated object O of $\mathcal{C}$,
2. for every object A an element $\varphi_{A,O} \in \mathrm{Mor}_{\mathcal{C}}(A, O)$ and
3. a group isomorphism $\tau : \mathrm{Mor}_{\mathcal{C}}(O, O) \xrightarrow{\simeq} \mathrm{Mor}_{\mathcal{G}}(*, *) \sim G$

Using these – non-canonically chosen – data we can define bijections $\alpha_{O,A} : \mathrm{Mor}_{\mathcal{C}}(X,O) \to \mathrm{Mor}_{\mathcal{C}}(O,O)$ by mapping $\varphi_{X,O} \mapsto \mathrm{id}_O$ and then extending this definition using post-composition with automorphisms of O. Using connectedness of the groupoid and invertibility of the morphisms, this gives rise to bijections $\alpha_{A,B} : \mathrm{Mor}_{\mathcal{C}}(A,B) \to \mathrm{Mor}_{\mathcal{C}}(O,O)$ for any pair of objects A, B by $\varphi_{B,O}^{-1}\varphi_{A,O} =: \varphi_{A,B}$.

Diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(B,C) \times \mathrm{Mor}_{\mathcal{C}}(A,B) \arrow[r, "\circ"] \arrow[d, "\alpha_{B,C}\times\alpha_{A,B}"'] & \mathrm{Mor}_{\mathcal{C}}(A,C)\ , \arrow[d, "\alpha_{A,C}"] \\
\mathrm{Mor}_{\mathcal{C}}(O,O) \times \mathrm{Mor}_{\mathcal{C}}(O,O) \arrow[r, "\mu"'] & \mathrm{Mor}_{\mathcal{C}}(O,O)
\end{tikzcd}$$

where μ is the multiplication in the automorphism group $\mathrm{Mor}_{\mathcal{G}}(O,O) \simeq G$, commute, ensuring that the definition is compatible with composition.

In this setup define functors

$$\begin{array}{llll}
F\colon \mathcal{C} \longrightarrow \mathcal{G} & A \longmapsto * & \mathrm{Mor}_{\mathcal{C}}(A,B) \ni f \longmapsto \tau\alpha_{A,B}(f) \\
H\colon \mathcal{G} \longrightarrow \mathcal{C} & * \longmapsto O & \mathrm{Mor}_{\mathcal{G}}(*,*) \ni g \longmapsto \tau^{-1}(g)
\end{array}$$

and show that there are natural isomorphisms $FH \xrightarrow{\sim} id_{\mathcal{G}}$ and $HF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ to prove equivalence of the categories $\mathcal{C}$ and $\mathcal{G}$. where $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{G}}$ are the identity functors on $\mathcal{C}$ and $\mathcal{G}$ respectively.

Diagrams

$$\begin{tikzcd}
* = & FH(*) \arrow[r, "FH(g)"] \arrow[d, "\mathrm{id}_*"'] & FH(*) \arrow[d, "\mathrm{id}_*"] & = * \\
* = & \mathrm{id}_{\mathcal{G}}(*) \arrow[r, "\mathrm{id}_{\mathcal{G}}(g)"'] & \mathrm{id}_{\mathcal{G}}(*) & = *
\end{tikzcd}$$

commute for any $g \in \mathrm{Mor}_{\mathcal{G}}(*,*)$ as $FH(g) = \tau\alpha_{O,O}\tau^{-1}(g) = g$ for $\alpha_{O,O} = \mathrm{id}_O$, proving the first direction. To show the second, consider

$$\begin{tikzcd}
O = & HF(A) \arrow[r, "HF(f)"] \arrow[d, "\vartheta_A"'] & HF(B) \arrow[d, "\vartheta_B"] & = O \\
X = & \mathrm{id}_{\mathcal{C}}(A) \arrow[r, "f"'] & \mathrm{id}_{\mathcal{C}}(B) & = O
\end{tikzcd} \tag{1}$$

where $\vartheta_C = \alpha_{O,C}^{-1}(\mathrm{id}_O)$. As we have for $f : A \to B$ a commutative diagram

$$\begin{tikzcd}
O \arrow[r, "\alpha_{A,B}(f)"] \arrow[d, "\varphi_{O,A}"'] & O \arrow[d, "\varphi_{O,B}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and $\varphi_{O,O} = \mathrm{id}_O$ it follows that

$$\begin{aligned} \vartheta_B HF(f) &= \left(\alpha^{-1}_{O,B}(\mathrm{id}_O)\right) \tau^{-1}\tau\alpha_{A,B} f \\ &= \alpha^{-1}_{O,B}\varphi^{-1}_{O,B} f \varphi_{O,A} \\ &= f\varphi_{O,A}, \end{aligned}$$

thus $f\vartheta_A = f\alpha^{-1}_{O,A}(\mathrm{id}_O) = f\varphi_{O,X}\mathrm{id}_O\varphi^{-1}_{O,O} = f\varphi_{O,X}$ and hence the diagram 1 commutes. As the $\vartheta_C = \alpha^{-1}_{O,C}(\mathrm{id}_O) = \varphi_{O,C}\varphi_{O,O}$ are invertible, the equivalence of categories is shown. □

Examples There are plenty of examples for groupoids. One would be the category with morphisms being paths in a space X up to reparametrisation where the source and target maps are evaluation at 0 and 1 respectively.

More examples and further discussion are given in the introductory text by Weinstein [2] and the more in-depth survey by Brown [1].

References

[1] Ronald Brown. From groups to groupoids: a brief survey. *Bull. London Math. Soc.*, 19(2):113–134, 1987.

[2] Alan Weinstein. Groupoids: unifying internal and external symmetry. A tour through some examples. *Notices Amer. Math. Soc.*, 43(7):744–752, 1996.