CLAIM A critical point x of $f: M \to \mathbb{R}$ is non-degenerate iff $df: M \to T^*M$ is transverse to the zero section at x;

PROOF First note that T^* is a functor:

$$\begin{aligned} T^*: \text{smooth manifolds} &\longrightarrow \text{smooth manifolds} \\ M &\longmapsto T^*M \\ (\varphi: M \to N) &\longmapsto T^*\varphi: T^*M \to T^*N \\ & \qquad (x, \alpha) \longmapsto (\varphi(x), \alpha \circ (d_x\varphi)^{-1}) \end{aligned}$$

$$\begin{tikzcd} T^*\mathcal{U} \arrow[r, "T^*\varphi"] & T^*\mathbb{R}^n \\ \mathcal{U} \arrow[u, "df"] \arrow[r, "\varphi"] & \mathbb{R}^n \arrow[u, "d(f\circ\varphi^{-1})"'] \end{tikzcd}$$

In particular, $T^*\varphi$ maps the zero-section on T^*M to the zero section on T^*N. Thus for $N = \mathbb{R}^n$, $M = U$ and φ a chart $\varphi: \mathcal{U} \xrightarrow{\sim} \mathbb{R}$ around x. By this, we have

$$df \pitchfork M \times \{0\}$$

$$\iff d(f \circ \varphi^{-1}) \pitchfork \mathbb{R}^n \times \{0\}$$

write $h := f \circ \varphi^{-1}$:
then $dh: \mathbb{R}^n \longrightarrow T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$
$\qquad x \longmapsto (x, \sum \partial h/\partial x_i|_x)$

$$\iff dh \pitchfork \mathbb{R}^n \times \{0\}$$

$$\iff \underset{\underset{\displaystyle\mathbb{R}^{2n}}{\wr\wr}}{T_{h(x)}(\mathbb{R}^n \times \mathbb{R}^n)} = d_x dh(T_x\mathbb{R}^n) + \underset{\underset{\displaystyle\mathbb{R}^{n}}{\wr\wr}}{T_{h(x)}\mathbb{R}^n}$$

$$\iff \sum \partial h/\partial x_i\Big|_x \text{ is a submersion}$$

$$\iff d\Big(\sum \partial h/\partial x_i|_x\Big) \text{ is non-singular}$$

$$\iff \operatorname{hess}_x(dh) \text{ is nonsingular.}$$

$$\iff x \text{ is a non-degenerate critical point of } h$$

$$\iff x \text{ is a non-degenerate critical point of } f.$$

$\square$