CONNECTIONS, CONNECTIONS, CONNECTIONS

ABSTRACT. Connections are the essential component of differential geometry. This text looks at a number of definitions for a connection and discusses the equivalences between them. An effort is made to present the workings of those equivalences in detail.

Disclaimer: This text was never finished. Section 6 is incomplete an further sections were meant to be added.

1. PRELIMINARIES

The following section will present various definitions of a connection and the equivalences between them. As an effort is made to present in precise detail, we require quite a bit of notation which is fixed here.

Let G be a Lie group with unit e and multiplication m. The multiplication gives rise to actions by left and right multiplication with an element g of G, l_g and r_g, as well as the adjoint action $\mathrm{Ad} = r_{g^{-1}} l_g$: $m, l_g, r_g, \mathrm{Ad}_g$

$$\begin{array}{llll} m : G \times G \to G & l_g : G \to G & r_g : G \to G & \mathrm{Ad}_g : G \to G \\ (g\ ,\ h) \mapsto gh & h \mapsto gh & h \mapsto gh & h \mapsto ghg^{-1} \end{array}$$

Taking derivates of these yields maps of the tangent bundle:

$$D_h l_g : T_h G \to T_{gh} G \qquad D_h r_g : T_h G \to T_{hg} G \qquad ad_g : T_h G \to T_h G$$

At $h = e$, the derivative of the adjoint action gives rise to the adjoint representation on the Lie algebra $\mathbf{g} = T_e G$ of G.

We define a $\mathbf{g}$-valued 1-form ϑ on G which uses left multiplication to push a tangent vector $X \in T_g G$ into the Lie algebra: ϑ

$$\vartheta_g(X) = D l_{g^{-1}}(X)$$

This is known as the *Maurer-Cartan form* on G.

Next, consider a smooth right action of G on a manifold P. It is given by a map R

$$R : P \times G \longrightarrow P \qquad (p, g) \longmapsto R(p, g) = p.g$$

which is compatible with the multiplication and unit in G, that is, the following two diagrams commute:

$$\begin{array}{ccc} P \times G \times G & \xrightarrow{R \times \mathrm{id}} & P \times G \\ {\scriptstyle \mathrm{id} \times m} \downarrow & & \downarrow {\scriptstyle R} \\ P \times G & \xrightarrow[R]{} & P \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\mathrm{id} \times e} & P \times G \\ & {\scriptstyle \mathrm{id}_P} \searrow & \downarrow {\scriptstyle R} \\ & & P \end{array}$$

This gives rise to multiplication maps for a fixed group element $g \in G$ or a fixed point $p \in p$: R_g, R^p

$$R_g : P \longrightarrow P \qquad p \longmapsto p.g \qquad\qquad R^p : G \longrightarrow P \qquad g \longmapsto p.g$$

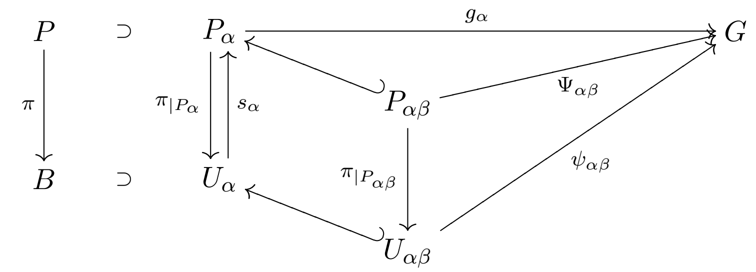

FIGURE 1. Maps we use when working with bundles.

Just as for the group multiplication, we consider derivatives of these maps:

$$D_pR_g : T_pP \longrightarrow T_{p.g}P \qquad\qquad D_gR^p : T_gG \longrightarrow T_{p.g}P$$

and we need these when differentiating R with respect to a variable x on which both factors of its argument depend as the Leibniz rule [Por07] has to be used:

$$D_x(R \circ (p, g)) = D_{g(x)}R_{p(x)}\, D_xg(x) + D_{p(x)}R^{g(x)}\, D_xp(x)$$

$\overline{Y}$ We also use R^p to define the *fundamental vector field* $\overline{Y}$ on P for a given element $Y \in \mathbf{g}$ as: $\overline{Y}_p = D_eR^p(Y)$. Note that for $P = G$ and $R = m$, we have that $R^g = l_g$. It follows that the Maurer-Cartan form evaluates a fundamental vector field to the element of the Lie algebra that generated it:

$$\vartheta(\overline{Y})_g = D_gl_{g^{-1}}\, D_eR^g(Y) = D_gl_{g^{-1}}\, D_el_g(Y) = Y.$$

A *right principal G-bundle* on a manifold B is a locally trivial fibre bundle
$\pi : P \to B$ $\pi : P \to B$ with a right G-action that is free and transitive on fibres. For a
U_α sufficiently fine open cover $\{U_\alpha\}_{\alpha\in A}$ of B with intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of its sets, the bundle is given by a family of *transition maps* $\{\psi_{\alpha\beta} : U_{\alpha\beta} \to G\}$
$\psi_{\alpha\beta}$ satisfying the cocycle condition $\psi_{\alpha\gamma} = \psi_{\beta\gamma}\psi_{\alpha\beta}$ on triple intersections. It will
$\Psi_{\alpha\beta}$ be convenient to use the notation $\Psi_{\alpha\beta} = \psi_{\alpha\beta}\pi$.

s A *section* of P is a family of maps $s = \{s_\alpha\}_{\alpha\in A}$ satisfying $s_\beta = s_\alpha.\psi_{\alpha\beta}$. Given a section s of P and a point $p \in P_\alpha = \pi^{-1}(U_\alpha)$, the composition $s_\alpha\pi(p)$ gives a point in the same fibre as p. By transitivity of the G-action there is a unique $g \in G$ such that $p = s_\alpha\pi(p).g$. This defines a family of
g_α equivariant maps $\{g_\alpha : P_\alpha \to G\}_{\alpha\in A}$ which satisfy $g_\beta = (\psi_{\alpha\beta}^{-1}\pi)g_\alpha$.

We shall also use the transition maps $\psi_{\alpha\beta}$ to pull back the Maurer-Cartan
$\vartheta_{\alpha\beta}$ form to $U_{\alpha\beta}$: $\vartheta_{\alpha\beta} = {\psi_{\alpha\beta}}^*\vartheta$.

This approach avoids using trivialisation isomorphisms of the bundle but uses transition maps instead. Both methods are equivalent as the choice of a section s of P made in this approach corresponds to fixing a trivialisation in the other one. The section s can the thought of as the unit section a trivialisation gives rise to.

2. HORIZONTAL SUBSPACES

We begin with the most classical definition of a connection. Given a point p in a principal G-bundle P, the tangent space of P at p has a subspace along the fibre $V_p = \ker(D_p\pi) \subset T_pP$. V_p is called the *vertical subspace* of T_pP and all of these together form the *vertical subbundle* $V \subset TP$. A connection then designates a smooth complement of V, called the *horizontal subbundle*.

Definition. *A* connection *on P is a distribution H in TP such that $H \oplus V = TP$ and H is G-invariant, that is, $H_{p.g} = R_{g*}H_p$ for all $p \in P$ and $g \in G$.* $H \oplus V = TP$ $H.g = H$

Thanks to the direct sum, tangent vectors can be written uniquely as a sum of their vertical and horizontal parts: $X = v(X) + h(X)$.

The horizontal distribution gives rise to an isomorphism $T_{\pi(p)}B \to H_p$ for all $p \in P$, which can be used to uniquely lift a vector field Z on M to a horizontal right-invariant vector field Y on P. Y is known as the *horizontal lift* of Z.

[KN63, chapter II] [Joy00, 2.1.3]

3. 1-FORM ON P

We move on to express a connection as a differential form on the total space of the G-bundle P.

Definition. *A* connection *on P is a* **g***-valued 1-form η on P satisfying $\eta(\overline{X}) = X$ and $R_g^*\eta = \mathrm{ad}_{g^{-1}}\eta$ for all $X \in \mathbf{g}$, $g \in G$.* $\eta \in \Omega^1(P, \mathbf{g})$ $\eta(\overline{X}) = X$ $R_g^*\eta = \mathrm{ad}_{g-1}\eta$

To see that this gives a connection in terms of horizontal subspaces, assume we have a connection form η and define the subbundle H with fibre $H_p = \ker \eta_p \subset T_pP$ at each $p \in P$. Given a tangent vector $Y \in T_pP$, define the vector $VY = \overline{\eta(Y)}$ which is vertical as it comes from the group action along the fibre. For the non-vertical part $Y - VY$ of Y we have $H \Leftarrow \eta$

$$\eta_p(Y - VY) = \eta_p(Y) - \eta_p(\overline{\eta(Y)}_p) = \eta_p(Y) - \eta_p(Y) = 0,$$

showing that $Y - VY \in H_p$ and thus that $H_p + V_p = T_pP$. As η_p is surjective by definition, this sum is direct for dimensional reasons. To see the G-invariance of the H_p note that for any $Y \in H_p$ equivariance of η gives

$$\eta(R_{g*}Y) = R_{g*}\eta(Y) = \mathrm{ad}_{g^{-1}}\eta(Y) = 0$$

and hence $R_{g*}H_p \subset H_{p.g}$ with equality holding by non-degeneracy of R_{g*}. Thus H is a connection.

Conversely, start with a connection H in terms of horizontal subspaces. Let $p \in P$, $Y \in T_pP$ and consider the vertical vector $v(Y)$ gotten by projecting Y along the direction of the connection's horizontal subspace. Define a **g**-valued 1-form η by requiring that $\overline{\eta_p(Y)}_p = v(Y)$. This implies that $\eta_p(\overline{X}) = X$ for all $X \in \mathbf{g}$ and $p \in P$. $H \Rightarrow \eta$

To show equivariance of η, note that for the horizontal part $h(Y)$ of a tangent vector Y both sides of the equation vanish:

$$R_g^*\eta(h(Y)) = \eta(R_{g*}h(Y)) = 0 = \mathrm{ad}_{g^{-1}}\eta(h(Y)).$$

The vertical part of Y is in the fundamental vector field $\overline{X}$ for a vector $X \in \mathbf{g}$. This makes $R_{g*}v(Y)$ a vector in the fundamental vector field for $\mathrm{ad}_{g^{-1}}X$, thus giving

$$(R_g^*\eta)_p(v(Y)) = \eta_{p.g}(R_{g*}v(Y)) = \mathrm{ad}_{g^{-1}}X = \mathrm{ad}_{g^{-1}}\eta_p(v(Y))$$

as desired. □

[KN63, II.1] [BC64, chapter 5]

4. LOCAL 1-FORMS ON B

The equivariance condition of the previous section's definition implies that knowing a connection form's behaviour at a single point $p \in P$ gives the form's behaviour on all of the fibre $P_{\pi(p)}$; meaning that the connection form on the total space P of the bundle can almost be captured by a form on the base space only. The caveat being that this only works for trivial bundles and that we require a family of compatibly defined *local* 1-forms on the base space for the trivialisation of a general bundle.

$\eta_a \in \Omega^1(U_\alpha, \mathbf{g})$
$\eta_\beta = \mathrm{ad}_{\psi_{\alpha\beta}^{-1}} \eta_\alpha + \vartheta_{\alpha\beta}$

Definition. *A* connection *on P is given by a family $\{\eta_\alpha\}_{\alpha\in A}$ of 1-forms $\eta_\alpha \in \Omega^1(U_\alpha, \mathbf{g})$ such that $\eta_\beta = \mathrm{ad}_{\psi_{\alpha\beta}^{-1}} \eta_\alpha + \vartheta_{\alpha\beta}$ on all $U_{\alpha\beta}$.*

$\eta \Rightarrow \eta_\alpha$

To see that a connection given as a $\mathbf{g}$-valued 1-form η on the total space gives rise to a connection according to this definition, define a family of local forms on the base space by pulling back the global form using a section s of P:

$$\eta_\alpha = {s_\alpha}^* \eta \tag{1}$$

We have to show that these satisfy the equation from the definition. That is, we have to compute $\eta \circ Ds_\alpha$ on double intersections. Let $b \in U_{\alpha\beta}$ be a base point and $Z \in T_b U_{\alpha\beta}$ a tangent vector. Then differentiating s_β with the Leibniz rule and denoting all the indices gives

$$\begin{aligned} D_b s_\beta(Z) &= D_b(R_{\psi_{\alpha\beta}(b)} s_\alpha)(Z) \\ &= D_{s_\alpha(b)} R_{\psi_{\alpha\beta}(b)} D_b s_\alpha(Z) + D_{\psi_{\alpha\beta}(b)} R^{s_\alpha(b)} D_b \psi_{\alpha\beta}(Z) \\ &= D_{s_\alpha(b)} R_{\psi_{\alpha\beta}(b)} D_b s_\alpha(Z) + D_e R^{s_\beta(b)} D_{\psi_{\alpha\beta}(b)} l_{\psi_{\alpha\beta}^{-1}(b)} D_b \psi_{\alpha\beta}(Z) \end{aligned}$$

as $s_\beta(b) = s_\alpha(b).\psi_{\alpha\beta}^{-1}(b) = p$ and with associativity of the group action it follows that

$$DR^{s_\beta(b)} = DR^{s_\alpha(b).\psi_{\alpha\beta}^{-1}(b)} = DR^{s_\alpha}(b) Dl_{\psi_{\alpha\beta}^{-1}(b)}.$$

We evaluate η on this expression. For the the first summand using equivariance of the connection form η gives:

$$\begin{aligned} \eta_p(D_{s_\alpha(b)} R_{\psi_{\alpha\beta}(b)}(D_b s_\alpha(Z))) &= ({R_{\psi_{\alpha\beta}(b)}}^* \eta)_{s_\alpha(b)}(D_b s_\alpha(Z)) \\ &= \mathrm{ad}_{\psi_{\alpha\beta}^{-1}(b)} \left(\eta_{s_\alpha(b)}(D_b s_\alpha(Z))\right) \\ &= \mathrm{ad}_{\psi_{\alpha\beta}^{-1}(b)} ({s_\alpha}^* \eta)_b(Z) \\ &= \mathrm{ad}_{\psi_{\alpha\beta}^{-1}(b)} (\eta_\alpha)_b(Z). \end{aligned}$$

For the second summand, observe that it is the value at p of the fundamental vector field $\overline{Y}$ for $Y = D_{\psi_{\alpha\beta}(b)} R_{\psi^{-1}_{\alpha\beta}(b)} D_b \psi_{\alpha\beta}(Z)$. And thus

$$\begin{aligned}\eta_p(D_e R^{s_\beta(b)} D_{\psi_{\alpha\beta}(b)} l_{\psi^{-1}_{\alpha\beta}(b)} D_b \psi_{\alpha\beta}(Z))&\\ &= D_{\psi_{\alpha\beta}(b)} l_{\psi^{-1}_{\alpha\beta}(b)} D_b \psi_{\alpha\beta}(Z)\\ &= ({\psi_{\alpha\beta}}^* \vartheta)_b(Z)\\ &= {\vartheta_{\alpha\beta}}_b(Z).\end{aligned}$$

It follows that $\eta_\beta = \eta\, Ds_\beta = \mathrm{ad}_{\psi^{-1}_{\alpha\beta}} \eta_\alpha + \vartheta_{\alpha\beta}$ holds as desired.

The other way round we need to construct a global connection form η on P from a connection given as a family of local 1-forms on B. Assume that such a connection form on P exists and that it is related to the given family of η_α by pulling back with a chosen section s of P (1). The following argument shows that these conditions uniquely determine η and give an explicit formula for it which we will then use. $\eta \Leftarrow \eta_\alpha$

Using the maps $g_\alpha : P_\alpha \to G$ we write the identity map on P_α as a fancy composition:

$$\mathrm{id}_{P_\alpha} : P_\alpha \xrightarrow{\quad\Delta\quad} P_\alpha \times P_\alpha \xrightarrow{s_\alpha\pi\times g_\alpha} P_\alpha \times G \xrightarrow{\quad R\quad} P_\alpha$$

Taking derivatives and not forgetting the Leibniz rule gives

$$\mathrm{id}_{T_pP_\alpha} = D_p \mathrm{id}_{P_\alpha} = D_{s\pi(p)} R_{g(p)}\, D_p s\pi + D_{g(p)} R^{s\pi(p)}\, D_p g_\alpha$$

At $p \in P_\alpha$ we re-write $\eta(Y)$ using equivariance

$$\begin{aligned}\eta_p(Y_p) &= \eta_p\left(D_{s_\alpha\pi(p)} R_{g_\alpha(p)}\, D_p s\pi(Y)\right) + \eta_p\left(D_{g_\alpha(p)} R^{s_\alpha\pi(p)}\, D_p g_\alpha(Y)\right)\\ &= \mathrm{ad}_{g_\alpha^{-1}(p)}\, \eta_{s\pi(p)}\left(D_p s\pi(Y)\right)\\ &\qquad + \eta_p\left(D_{g_\alpha(p)} R^{s_\alpha\pi(p)}\, D_e l_{g_\alpha(p)}\, D_{g_\alpha(p)} l_{g_\alpha^{-1}(p)}\, D_p g_\alpha(Y)\right),\end{aligned}$$

where the identity $Dl_{g_\alpha(p)}\, Dl_{g_\alpha^{-1}(p)}$ was inserted in the last summand. This amounts to the whole argument of η_p in that summand being a fundamental vector field at $s_\alpha\pi(p).g_\alpha(p) = p$ which is generated by $Dl_{g_\alpha^{-1}(p)}\, Dg(Y)$. It follows that, on P_α, η has to be of the form

$$(\eta_{|P_\alpha})_p = \mathrm{ad}_{g_\alpha^{-1}(p)}\, \eta_{\alpha\pi(p)}\, D\pi + D_{g_\alpha(p)} l_{g_\alpha^{-1}(p)}\, D_p g_\alpha$$

which we shall use to define η on each P_α. It remains to check that this is well-defined, equivariant and evaluates fundamental vector fields correctly.

To see *well-definedness*, let $p \in P_{\alpha\beta}$, simplify notation by omitting the points of differentiation and write $\Psi_{\alpha\beta} = \psi_{\alpha\beta}\pi$ to compute

$$\begin{aligned}(\eta_{|P_\beta})_p &= \mathrm{ad}_{g_\beta^{-1}(p)}\, \eta_\beta\, D\pi + Dl_{g_\beta^{-1}(p)}\, Dg_\beta\\ &= \mathrm{ad}_{g_\alpha^{-1}(p)\Psi_{\alpha\beta}(p)}\, \mathrm{ad}_{\Psi^{-1}_{\alpha\beta}(p)}\, \eta_\alpha\, D\pi + \mathrm{ad}_{g_\alpha^{-1}(p)\Psi_{\alpha\beta}(p)}\, Dl_{\Psi^{-1}_{\alpha\beta}(p)}\, D\Psi_{\alpha\beta}\\ &\quad + Dl_{g_\alpha^{-1}(p)}\, Dl_{\Psi_{\alpha\beta}(p)}\, Dl_{\Psi^{-1}_{\alpha\beta}(p)}\, Dg_\alpha + Dl_{g_\alpha^{-1}(p)}\, Dl_{\Psi_{\alpha\beta}(p)}\, Dr_{g_\alpha(p)}\, D\Psi^{-1}_{\alpha\beta}.\end{aligned}$$

After cancellations the first and the third summand give exactly $(\eta_{|P_\alpha})_p$ while the remaining two summands cancel out as again the Leibniz rule gives for the constant unit map $e : P_{\alpha\beta} \to G$

$$0 = De = D(\Psi_{\alpha\beta}\Psi^{-1}_{\alpha\beta}) = Dl_{\Psi_{\alpha\beta}}\, D\Psi^{-1}_{\alpha\beta} + Dr_{\Psi^{-1}_{\alpha\beta}}\, D\Psi_{\alpha\beta}$$

and thus, expanding the adjoint and using that Dl_g and Dr_g commute gives

$$\begin{aligned}\mathrm{ad}_{g_\alpha^{-1}\Psi_{\alpha\beta}}\, Dl_{\Psi_{\alpha\beta}^{-1}}\, D\Psi_{\alpha\beta} &= Dl_{g_\alpha^{-1}}\, Dl_{\Psi_{\alpha\beta}}\, Dr_{g_\alpha}\, Dr_{\Psi_{\alpha\beta}^{-1}} Dl_{\Psi_{\alpha\beta}^{-1}} D\Psi_{\alpha\beta} \\ &= -Dl_{g_\alpha^{-1}}\, Dl_{\Psi_{\alpha\beta}}\, Dr_{g_\alpha}\, D\Psi_{\alpha\beta}^{-1},\end{aligned}$$

which is just what we need for the terms to cancel.

Equivariance of the form follows from equivariance of g_α. For any $h \in G$ and $p \in P_\alpha$ we have:

$$\begin{aligned}{R_h}^* \eta_p &= \eta_{p.h}\, DR_h \\ &= \mathrm{ad}_{g_\alpha^{-1}(p.h)}\, \eta_\alpha\, D\pi + Dl_{g_\alpha^{-1}(p.h)}\, Dg_\alpha\, DR_h \\ &= \mathrm{ad}_{h^{-1}}\, \mathrm{ad}_{g_\alpha^{-1}}\, \eta_\alpha\, D\pi + Dl_{h^{-1}} Dl_{g_\alpha^{-1}(p)}\, Dr_h\, Dg_\alpha \\ &= \mathrm{ad}_{h^{-1}}\, (\mathrm{ad}_{g_\alpha^{-1}}\, \eta_\alpha\, D\pi + Dl_{g_\alpha^{-1}(p)}\, Dg_\alpha) \\ &= \mathrm{ad}_{h^{-1}}\, \eta_p\end{aligned}$$

To investigate the *behaviour on fundamental vector fields* note that $g_\alpha \circ R^p = l_{g_\alpha(p)}$ and use definitions. With $p \in P$, $X \in \mathbf{g}$ and $\overline{X}$ being vertical, it follows that

$$\begin{aligned}\eta(\overline{X})_p &= \mathrm{ad}_{g_\alpha^{-1}(p)}\, \eta_\alpha\, D\pi(\overline{X}) + Dl_{g_\alpha^{-1}(p)}\, Dg\, D_e R^p(X) \\ &= 0 + Dl_{g_\alpha^{-1}(p)}\, Dl_{g_\alpha(p)} X = X\end{aligned}$$

and hence we recover a connection form on P from a local connection form on B. □

Both constructions made in this section depend on the choice of the section s. In that sense the construction is not canonical.

[KN63, II.1] [Oku87, 2] [Wak71, II.8]

5. PARALLEL TRANSPORT AND HOLONOMY

The definition of a connection in terms of horizontal subspaces in §2 is motivated by the idea of parallel transport: Given a point $p \in P$ and a piecewise smooth path $\gamma : [0, 1] \to B$ with $\pi(p) = \gamma(0)$, one can ask how the position of p in the fibre changes when its basepoint moves along γ. A connection H is designed to determine just that.

We can easily lift the path γ in B to *some* path $\tilde{\Gamma}$ in P starting at p. However, that path is neither unique nor necessarily *horizontal* in the sense that its tangent vectors are in the horizontal subbundle. Yet it is useful to start with such an arbitrary lift as we can express the difference between it and the horizontal lift Γ we want using a path g in G that starts at the unit and the G-action on the bundle:

$$\Gamma(t) = \tilde{\Gamma}(t).g(t)$$

Tangent vectors to that path are expressed using the Leibniz rule

$$\begin{aligned}D_t\Gamma &= D_{\tilde{\Gamma}(t)} R_{g(t)}\, D_t\tilde{\Gamma} + D_{g(t)} R^{\tilde{\Gamma}(t)}\, D_t g \\ &= D_{\tilde{\Gamma}(t)} R_{g(t)}\, D_t\tilde{\Gamma} + D_e R^{\tilde{\Gamma}(t)}\, D_{g(t)} l_{g(t)^{-1}}\, D_t g\end{aligned}$$

where the last summand is a fundamental vector field at $\tilde{\Gamma}(t)$. Applying the differential form η arising for the connection by §3 to this gives

$$\begin{aligned}\eta(D_t\Gamma) &= \mathrm{ad}_{g(t)^{-1}}\,\eta(D_t\tilde{\Gamma}) + \eta(\overline{D_{g(t)}l_{g(t)^{-1}}\,D_tg}_{\tilde{\Gamma}(t)})\\ &= \mathrm{ad}_{g(t)^{-1}}\,\eta(D_t\tilde{\Gamma}) + D_{g(t)}l_{g(t)^{-1}}\,D_tg\end{aligned}$$

The tangent vector at the path Γ is horizontal if and only if this expression vanishes, that is, if and only if

$$Dr_{g(t)^{-1}}\,\eta(D_t\tilde{\Gamma}) = D_tg$$

This type of differential equation locally has a unique solution which can be extended to one on the full length of the path and thus gives us a unique horizontal lift Γ_p of γ with start point p. Applying the same argument for all possible start points p in the fibre above $\gamma(0)$, we define a map

$$PT_\gamma : P_{\gamma(0)} \longrightarrow P_{\gamma(1)} \qquad\qquad p \longmapsto \Gamma_p(1)$$

which is the *parallel transport* along γ. In particular PT_γ is G-equivariant and hence it is a diffeomorphism. In case γ is a loop, PT_γ is a loop, this makes PT_γ a translation of the fibre and thus corresponds to an element g of G. The subset

$$\mathrm{Hol}_b(H) = \{g \mid \exists \text{ a loop } \gamma \text{ with } \gamma(0) = \gamma(1) = b \text{ such that } PT_\gamma = r_g\} \subset G$$

is in fact a subgroup as composition of loops gives rise to two consecutive translations, i.e. multiplication, and reverting the direction of a loop gives the inverse element under PT. This subgroup is known as the *holonomoy group for H at b*.

For path-connected B any two points can be joined by a path and it follows that the isomorphism class of $\mathrm{Hol}_b(H)$ is independent of the point b and a change of point yields a subgroup of G which is isomorphic by conjugation. Hence the base point is frequently omitted from the notation and we speak of just the *holonomy group* $\mathrm{Hol}(H)$ of H.

While the holonomy group captures the essential information about a connection, we dropped too much information on the way (the base point of the loops, the map $\gamma \mapsto PT_\gamma$ being far from injective) to recover the exact connection we started with from just a subgroup of G. A more careful and technical approach has to be taken to preserve enough information. A sketch of the necessary steps along with a definition of a connection along these lines is given in what follows. The idea behind this isn't brand new and essentially can be found in the paper of Teleman [Tel69]. A full account of the construction closer to this presentation can be found in the papers of Caetano and Picken [CP94] and Barrett [Bar91].

The first idea to improve the starting point would be to not just keep the holonomy group as data but to take into account all the information of the map $\gamma \mapsto PT_\gamma$. This information, however, is highly redundant as we know that a reparametrisation of a path gives rise to the same parallel transport map as the original path. Likewise, inserting another path into a path and then inserting the same path in reverse orientation will not change the parallel transport map either.

We want to identify all paths giving rise to the same parallel transport map for such 'trivial' reasons. Roughly speaking, we will identify two paths with identical start end end points whenever the loop they form encloses an area that is at most one dimensional. For things to work smoothly we don't use ordinary paths but define the set of paths which are constant on a positive interval near their beginning and end. We can compose two such paths in the usual way by first running through one at twice the speed and then running through the other at twice the speed. We can also revert a path by the diffeomorphism $t \mapsto 1 - t$ of $[0, 1]$.

Definition. *An* intimacy *of two paths γ, δ in B is a smooth homotopy $h : [0, 1] \times [0, 1] \to B$ such that*

1. *at all* $(t, u) \in [0, 1] \times [0, 1] \mathrm{rank}(D_{(t,u)}H) \leq 1$
2. *there is an* $\epsilon \in [0, 1/2]$ *such that*

$$\begin{aligned} H(s,t) &= \gamma(t) \quad &&\textit{for} \quad s < \epsilon \\ H(s,t) &= \delta(t) \quad &&\textit{for} \quad s > 1 - \epsilon \\ H(s,t) &= \gamma(0) \quad &&\textit{for} \quad t < \epsilon \\ H(s,t) &= \delta(0) \quad &&\textit{for} \quad t > 1 - \epsilon \end{aligned}$$

It can be checked that intimacy gives an equivalence relation on the set of paths which are constant at the beginning and end. The relation is compatible with composing and inverting paths. We can consider the subset of all loops $\Omega(B)$ and its quotient $L(B)$ under the intimacy relation. For path-connected B note that every class has a representative based at an arbitrary point $b \in B$. Hence we can assume all elements of $L(B)$ to be based at the same point b. The homotopy aspect of the relation makes composition associative on the quotient, which makes composition in $L(B)$ a group operation.

Next, observe that we can use the intimacy h to pull back the connection form η on P to $[0, 1] \times [0, 1]$. The pulled-back connection has to be flat because the pulled-back curvature form is bound to be 0 as all the pairs of vectors we give it are linearly depend, for they are in the image of DH which is of rank 1 by the definition of an intimacy. It follows that the pulled-back bundle and the pulled-back connection are trivial and thus that parallel transport around $\partial([0, 1] \times [0, 1])$ gives the identity map. As that boundary is the composition of γ, the constant path at $\gamma(1)$, δ^{-1} and the constant path at $\gamma(0)$, it follows that the parallel transport maps we get for intimate γ and δ are the same.

With all this in place, we see that a connection H gives a group morphism $PT_H : L(B) \to G$. This inspires the following definition:

Definition. *A* connection *is a group morphism $PT_H : L(B) \mapsto G$ such that for any open subset $U \subset \mathbf{R}^n$ and any smooth map $\lambda : U \to \Omega(B)$ for which* $\mathrm{ev} \circ (\lambda \times \mathrm{id}_{[0,1]}) : U \times [0, 1] \to B$ *is smooth the composition*

$$U \xrightarrow{\lambda} \Omega(M) \xrightarrow{p} L(M) \xrightarrow{h} G$$

is smooth as well.

The preceding paragraphs indicated how we arrive at a connection in this sense when starting with a connection in terms of horizontal subspaces or a differential form on P. As this notion of a connection doesn't explicitly include the bundle we are using, going back to the original has to include reconstruction of the bundle the connection is defined on. A rough *sketch* of the reconstructions of the bundle and the connection is given in the following paragraphs.

The bundle is reconstructed as the set $P = (BP(B) \times G)/\sim$ where BP denotes based paths and the equivalence relation $(\gamma, g) \sim (\gamma', g')$ holds if and only if $\gamma(1) = \gamma'(1)$ and $PT_H(\gamma'\gamma^{-1})g = g'$. This P has a projection map $\pi : P \to B$, $[\gamma, g] \mapsto \gamma(1)$ and there is a free right G-action with $R_h([\gamma, g]) = [\gamma, gh]$.

A trivialisation of this bundle is constructed as follows: Using an atlas for B whose charts t_α are in the unit ball, we can find paths γ_α^m such that $\gamma_\alpha^m(0) = t_\alpha^{-1}(0)$ and $\gamma_\alpha^m(1) = m$ where γ_α^m is smooth, constant near 0 and 1 and maps to a straight line in the chart t_α. Fixing a path $p \in BP(B)$ with $p(1) = t^{-1}(0)$ that is constant near 0 and 1, note that any element of P can be written as $[p\gamma_\alpha^m, g]$. Using that we can define maps $\varphi_\alpha : [p\gamma_\alpha^m, g] \mapsto (m, g)$ which give a trivialisation of P.

Comparing two trivialisation maps for different charts leads to transition maps $\psi_{\alpha\beta} : m \mapsto PT_H(p_\beta\gamma_\beta^m\gamma_\alpha^{m-1}p_\alpha^{-1})$ whose smoothness is ensured by the smoothness condition of the composition in the definition of a connection in this section.

To reconstruct the connection, paths $p \in BP(B)$ are lifted to paths in P with a given $g \in G$ as $\hat{p}_g : k \mapsto [p(k\,\cdot), g]$, which are smooth. A careful computation shows that the differential $D_t\hat{p}_g$ of the lifted path is completely determined by the differential D_tp of the original, meaning the lifting of paths gives a section $\Gamma : TB \to TP$. The image of this map is a smooth horizontal distribution which is right invariant and thus a connection.

This completes our round trip which omitted many details. Those can be found in the papers [CP94] or [Bar91].

[Spi70, II.8] [KN63, II.4] [CP94] [Bar91]

6. VECTOR BUNDLES

If, instead of a G-principal bundle, we consider a vector bundle E over M, we can make the following definition:

Definition. *A* connection *on E is a map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$ which is linear and satisfies the Leibniz condition $\nabla(fs) = f\nabla(s) + s \otimes df$ for all functions f on M and sections s of E.*

This is related to the notion of connection we had for principal bundles by keeping in mind that a rank k vector bundle E along with a k-dimensional representation ρ of a group G gives us a G-principal bundle P associated to E.

for each vector bundle and a given representation $\rho : G \to \mathrm{Aut}(V)$ of G on a c

[MT97, chapter 17] [GH78, 0.5] [Joy00, 2.1.2] [MS74, appendix C]

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