THE COHOMOLOGY OF ΩS^1

$H^* \Omega S^1$ is $\mathbb{R}^{\mathbb{Z}}$, all of which lives in degree 0.

We get $\mathbb{R}^{\mathbb{Z}}$ as ΩS^1 has connected components consisting of the maps $S^1 \longrightarrow S^1$ of degree $k \in \mathbb{Z}$. Furthermore, each connected component is contractible as any loop of degree k can be „straightened out" and reparamaterised continuously to give the standard degree k map $z \longmapsto z^k$.

Alternatively the fact can be proved as follows: ΩS^1 has components ordered by the degree of the loops. Consider a map

$$\mathbb{Z} = \coprod_{\mathbb{Z}} * \longrightarrow \Omega S^1$$

[illegible]

This induces homomorphisms on homotopy groups

$$\pi_k(\coprod *) \longrightarrow \pi_k(\Omega S^1) = \pi_{k+1}(S^1)$$

The induced map is an isomorphism in degree 0 (mapping path components to path components and also an isomorphism in all other degrees as $\pi_k(\coprod_{\mathbb{Z}} *) = 0 = \pi_k(S^1)$ for $k > 0$. Thus the original map is a weak homotopy equivalence and (as both S^1 is a CW-complex and the loop space of a CW-complex is homotopy equivalent to a CW-complex) hence a homotopy equivalence, proving the statement about cohomology.

[The Serre Spectral sequence cannot be used here as it only works for simply connected base spaces